

The Butterfly Problem

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In his article, “The Metamorphosis of the Butterfly Problem,”¹ Leon Bankoff presented a series of proofs and a brief history of what has become known as the *Butterfly Problem*. This very simple arrangement of a circle and five chords has been around the mathematical world for at least the better part of the last two centuries. Through the years, many mathematicians have tackled the Butterfly Problem producing at least thirteen distinct and conceptually different proofs of it, which Bankoff summarized in his article.² He suggested that this great interest in the Butterfly Problem may come from its initial visual simplicity and its resemblance to the insect (from which the problem’s name comes) which people find almost universally beautiful.³

Bankoff states the Butterfly Problem explicitly (quoted from a proposal in a late 1950s edition of *School Science and Mathematics*) as:

*In a circle (O), P is the midpoint of chord AB . Chords RS and TV pass through the point P . RV cuts AP at a point M , and ST cuts PB at a point N .
Prove ... that MP equals PN .*

Figure 1 (on the next page) illustrates this construction.

Mathematicians have approached the Butterfly Problem from many different mathematical angles, including several elementary Euclidean geometric concepts,

¹ Bankoff, Leon,
“The Metamorphosis of the Butterfly Problem,”
Mathematics Magazine, Volume 60, Number 4 (October, 1987), pages 195 to 210.
Accessed through the JSTOR archive.

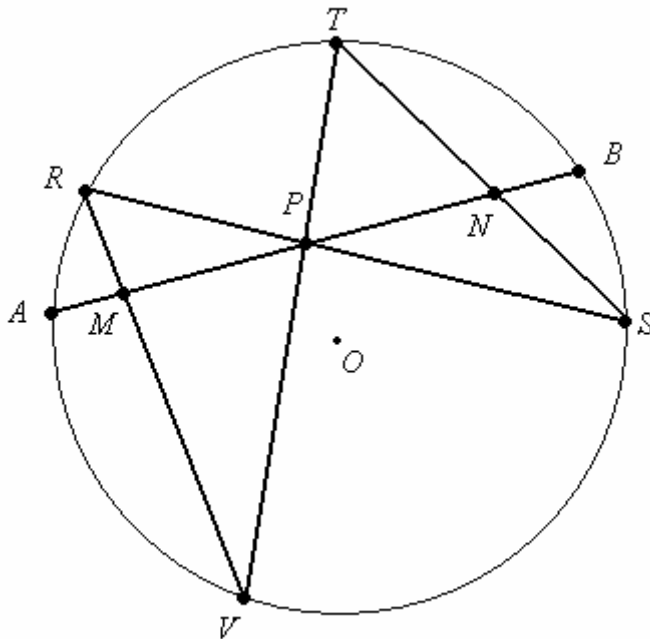
² Bankoff, page 195.

³ Bankoff, page 195.

trigonometry, analytic geometry, geometric transversal theory, and other methods of treatment. All of this study resulted in a wealth of material proving the Butterfly Problem in most any desired way.

The earliest two proofs of the Butterfly Problem that Bankoff found appeared in an 1812 edition of *The Gentleman's Diary* of London. They take varied, but equally sound approaches to the problem, and the second even generalizes the positions of some of the points in the Butterfly Problem.

Figure 1

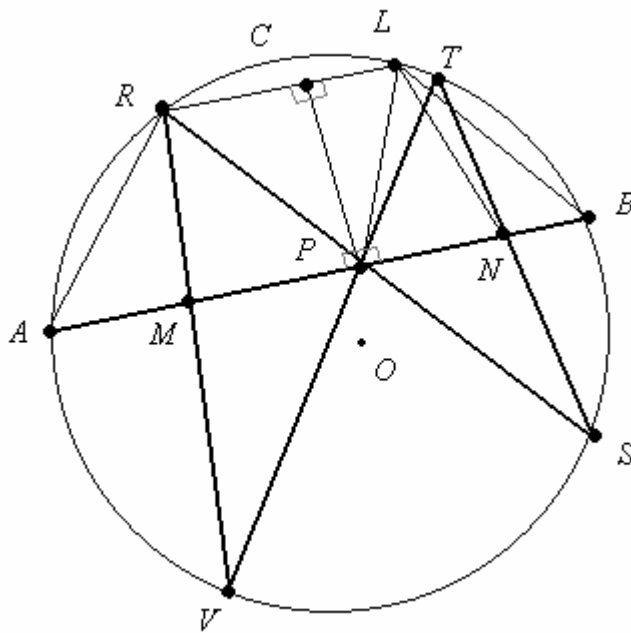


A brief study of the proofs that Bankoff presented confirms, as he suggested, that the most elegant proofs of this problem do not lie in the simplest mathematical approaches to solving it. The elementary Euclidean proofs are complex in setup and

somewhat cumbersome to follow,⁴ whereas the more advanced proofs, though more complicated in material, are cleaner in presentation.

The first proof that I will present, Bankoff took from his own article published in the February 1955 edition of *School Science and Mathematics*.⁵

Figure 2



⁴ See, for example, proofs number four and number six in Bankoff's article.

⁵ Bankoff, Leon

"Solution to Problem 2426,"

School Science and Mathematics, Volume 55 (1955), Page 156.

Proof 1: ⁶

See illustration in Figure 2, above.

To the basic construction, add chord RL, parallel to AB. Then make the segments LP, LN, RA, and LB.

Then, R and L are points equidistant from the center of RL. The center of RL is on the perpendicular from P, the midpoint of AB, since RL is parallel to AB and they are inscribed in a circle. Then, by Euclid's first theorem,⁷ triangles PCR and PCL are congruent. Thus, RP is equal to PL (so R and L are equidistant from P) as they are corresponding parts of congruent triangles PCR and PCL. Therefore, triangle RPL is isosceles, so angle PRL is congruent to angle PLR.

Next, triangle PRA is congruent to triangle PLB since they have two sides and the included angle congruent ($R \rightarrow P \rightarrow A$ and $L \rightarrow P \rightarrow B$, respectively). Also, angle LRS and angle LTS are supplementary as part of inscribed quadrilateral RLTS by Euclid's eighteenth theorem.⁸ Since angle LRS, angle RLP, and angle LPN are congruent (angle LPN by Euclid's twenty-fifth theorem⁹), and angle PRL (congruent to angle LPN) and angle LTS (angle LTN) are supplementary from RLTS, so angle LTS is supplementary to angle LPN, thus angle PNT is supplementary to angle PLT (as the sum of the interior angles of any quadrilateral is 360° and we have already accounted for 180°).

⁶ This proof corresponds to proof 1 in Bankoff's article.

⁷ Euclid's first theorem: "If two triangles have two sides and the included angle of one congruent, respectively, to two sides and the included angle of the other, then the two triangles are congruent." (SAS) (Smart, page 410)

⁸ Euclid's eighteenth theorem: "If a quadrilateral is inscribed in a circle, then the opposite angles are supplementary." (Smart, page 411)

⁹ Euclid's twenty-fifth theorem: "If two parallel lines are cut by a transversal, then the alternate interior angles are congruent and the corresponding angles are congruent." (Smart, page 411)

Therefore, by Theorem 4.8 in Smart,¹⁰ quadrilateral PNTL may also be inscribed in a circle. Hence, angle PLN is congruent to angle PTN by Euclid's twenty-second theorem.¹¹ Also, angle VTS (angle PTN) and angle SRV (angle PRM) are congruent by Euclid's twenty-second theorem. These two pairs of angles are congruent to each other because angle PTN is congruent to angle VTS. Thus, by Euclid's second theorem,¹² triangle MPR and triangle PLN are congruent. Therefore, the corresponding segments MP and PN are congruent.

Q.E.D.

Bankoff stated that this proof is also applicable to the case when the arc AR is smaller than the arc TR, so L lies between T and B in Figure 2, by substituting the word equal for the word supplementary in each instance of his outline. This method is not directly applicable to my expansion, as the words are used in more places than in the outline provided, but provides a good start to seeing where the proof's adaptation would go.

Now that we have seen a solution by elementary Euclidean geometry, let us look at a proof using analytic geometry. This method, Bankoff notes, may be approached in two ways: one rather harsh and one more reasonable. The harsher method begins by setting the x-axis along AB and the y-axis along MO. Then one would find the equation

¹⁰ Theorem 4.8 states: "A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary." (Smart, page 161)

¹¹ Euclid's twenty-second theorem: "An inscribed angle is measured by half its intercepted arc." (Smart, page 411)

¹² Euclid's second theorem: "If two triangles have two angles and the included side of one congruent, respectively, to two angles and the included side of the other, then the two triangles are congruent" (ASA) (Smart, page 410)

First, call the circle Γ_1 and define the plane of x and y coordinates with the x -axis coincident with line AB and the y -axis coincident with OM (perpendicular to AB since O and M are the midpoints). Then call the circle $O(0, d)$ where d is the height of O , so the point O is located at $(0, d)$ relative to the origin M.

If we define the circle's radius as r , its equation is $\sum_1 \equiv x^2 + (y - d)^2 - r^2 = 0$. Since CD and EF pass through M, they form a degenerate conic,¹⁶ Γ_2 , with equation $\sum_2 \equiv ax^2 + 2hxy + by^2 = 0$. Then let \sum be a conic formed by any linear combination of \sum_1 and \sum_2 , with equation $\sum \equiv l\sum_1 + m\sum_2 = 0$. Thus, \sum passes through all of the common points of Γ_1 and Γ_2 , in this case, points C, D, E, and F. Also, as both equations are general, all conics through these four points may be represented in the form of \sum for some l and m .

Suppose that a conic represented by \sum intersects AB (the x -axis) at two points, V and W. These points lie on the x -axis, so they are on the line $y = 0$. Thus, \sum_1 and \sum_2 reduce to $\sum_1(x, 0) \equiv x^2 + d^2 - r^2 = 0$ and $\sum_2(x, 0) \equiv ax^2 = 0$, respectively. Then, the roots of $\sum(x, 0) = 0$ are the abscissas¹⁷ of V and W, with $\sum(x, 0) \equiv k(x^2 + d^2 - r^2) + l(ax^2) = 0$, since these are the intersections with the x -axis.

Since all of the terms in $\sum(x, 0)$ are second order, its roots are symmetric about zero, so the sum of the roots is zero. Therefore, $MV + MW = 0$, yielding the result: $MV = WM$ (note the order of the endpoints). This solution is general, for all conics through points C, D, E, and F. Our original circle, Γ_1 , is a conic fitting these criteria, with lines

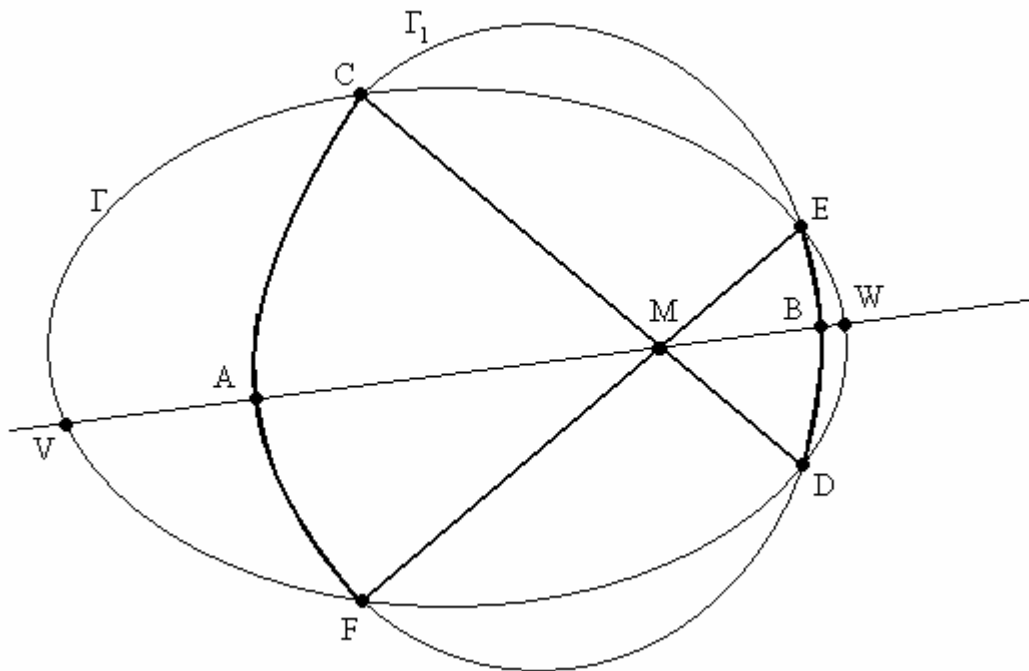
¹⁶ A degenerate conic is formed by the intersection of a double cone and a plane through the apex (center point) of the double cone. The result is a point a line or a pair of lines intersecting at the apex of the double cone.

¹⁷ The abscissas are the x coordinates of points and intersections. Here, they are the x coordinates of points V and W.

ED and CF. So, from the last pair of equations, we have $PM + QM = 0$, and therefore,
 $PM = MQ$.

Q.E.D.

Figure 4



These proofs are but two of many ways that have been used to solve the Butterfly Problem through the years. The remaining eleven in Bankoff's article involve several other geometric approaches and a few from other disciplines. Some of them also deal with related theorems and, more relevantly, with generalizations of the Butterfly Problem to varying positions of M and other similar constructions. I recommend a study of the remaining proofs for an in-depth, though daunting, look at this problem.

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